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Canonical solution of the state labelling problem for SU(n) ⊃ SO(n) and Littlewood's branching rule: II. Use of modification rules

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Abstract. In the first paper in the present series, it was shown that when $d \leq [\frac{1}{2}n]$, the internal state labelling problem for the d-row irreducible representations of SU(n), when reduced with respect to SO(n), amounts to the external state labelling problem for U(d). In this paper, this result is extended to the $d > [\frac{1}{2}n]$ case, where Littlewood's branching rule for U(n) ⊃ O(n) has to be supplemented with Newell's modification rules. The explicit solution of the state labelling problem for $(\nu+1)$-row irreducible representations of both SU($2\nu+1$) and SU(2ν) is given and is shown to reflect the operation of Littlewood's modified branching rule in a direct way. The generalisation to higher d values is then outlined.

1. Introduction

The purpose of this series of papers is to present a new solution of the state labelling problem for the d-row irreducible representations (irreps) of SU(n), when reduced with respect to SO(n). This solution, which is not restricted to small values of n or d, reflects in a direct way the operation of Littlewood's branching rule (1950) for the chain U(n) ⊃ O(n). Its derivation makes use of the complementarity relationship between the latter and the chain Sp($2d$, R) ⊃ U(d) for d-row irreps of U(n) (Moshinsky 1963, Moshinsky and Quesne 1970, 1971).

The first paper in this series (henceforth referred to as I and whose equations will be subsequently quoted by their number preceded by I) was devoted to the general formulation of the method (Deenen and Quesne 1983). For such purposes, only irreps of U(n) with no more than $\nu = [\frac{1}{2}n]$ rows were considered, thereby avoiding the difficulties involved in the conversion of non-standard symbols of O(n) into standard ones. The present paper removes this limitation and deals with the case where $d > \nu$. Littlewood's branching rule for U(n) ⊃ O(n) must then be supplemented with Newell's modification rules (1951). The present paper's purpose is to show that when $d > \nu$, the state labelling problem proposed solution is directly connected with Littlewood's modified branching rule.

In § 2, the state labelling problems for the complementary chains U(n) ⊃ O(n) and Sp($2d$, R) ⊃ U(d) are discussed in the case where $d > \nu$. The construction of highest weight states (HWS) of equivalent O(n) irreps is reviewed in § 3. In §§ 4 and 5, the

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particular cases of $(\nu+1)$-row irreps of $\mathrm{U}(2\nu+1)$ and $\mathrm{U}(2\nu)$ are respectively analysed in detail. Finally in § 6, the generalisation to higher d values is outlined.

2. The state labelling problems for $\mathrm{U}(n) \supset \mathrm{O}(n)$ and $\mathrm{Sp}(2d, R) \supset \mathrm{U}(d)$ when $d > \nu$

As mentioned in I, the discussion of the state labelling problem for $\mathrm{SU}(n) \supset \mathrm{SO}(n)$ reduces to the same discussion for either complementary group chain

$$\begin{array}{cc} \mathrm{U}(n) & \mathrm{O}(n), \\ [h_1 h_2 \ldots h_d] & (\lambda_1 \lambda_2 \ldots \lambda_\nu) \end{array} \tag{2.1}$$

$$\begin{array}{ccc} \mathrm{Sp}(2d, R) & \supset & \mathrm{U}(d). \\ \langle (\tfrac{1}{2}n)^{d-\nu}, \lambda_\nu + \tfrac{1}{2}n, \ldots, \lambda_1 + \tfrac{1}{2}n \rangle & & [h_1 h_2 \ldots h_d] \end{array} \tag{2.2}$$

Both pairs of groups $\mathrm{U}(n)$, $\mathrm{U}(d)$ and $\mathrm{O}(n)$, $\mathrm{Sp}(2d, R)$ are complementary (Moshinsky and Quesne 1970) within the irrep $\langle (\frac{1}{2})^{dn} \rangle$ or $\langle (\frac{1}{2})^{dn-1}\frac{3}{2} \rangle$ of a larger group $\mathrm{Sp}(2dn, R)$ (Moshinsky 1963, Moshinsky and Quesne 1971). Presently, we are interested in d-row irreps of $\mathrm{U}(n)$, such that $\nu < d < n$ and $n = 2\nu$ or $2\nu+1$. In equations (2.1) and (2.2), the quantum numbers characterising its irreps are indicated below each group.

The reduction $\mathrm{U}(n) \supset \mathrm{O}(n)$ is governed by Littlewood's branching rule (1950), provided the latter partitions $(\lambda_1 \lambda_2 \ldots \lambda_d)$ containing more than ν parts are converted into partitions containing no more than ν parts by using Newell's modification rules (1951). Since Littlewood's modified branching rule rapidly becomes very complicated when $d-\nu$ increases, we shall hereafter only quote it for the case corresponding to the smallest d value, namely $d = \nu+1$. When $n = 2\nu+1$, it can be written as

$$\begin{aligned} [h_1 \ldots h_{\nu+1}] = {} & \sum_{\lambda_1 \ldots \lambda_\nu}{}' \left(\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_\nu][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]} \right) (\lambda_1 \ldots \lambda_\nu) \\ & + \sum_{\lambda_1 \ldots \lambda_\nu \geqslant 1}{}'' \left(\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_\nu 1][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]} \right) (\lambda_1 \ldots \lambda_\nu), \end{aligned} \tag{2.3}$$

and when $n = 2\nu$ as

$$\begin{aligned} [h_1 \ldots h_{\nu+1}] = {} & \sum_{\lambda_1 \ldots \lambda_{\nu-2}} \left(\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_{\nu-2}][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]} \right) (\lambda_1 \ldots \lambda_{\nu-2}) \\ & + \sum_{\lambda_1 \ldots \lambda_{\nu-1} \geqslant 1} \left(\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_{\nu-1}][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]} \right. \\ & \left. + \sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_{\nu-1} 1^2][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]} \right) (\lambda_1 \ldots \lambda_{\nu-1}) \\ & + \sum_{\lambda_1 \ldots \lambda_{\nu-1} \geqslant 1} \left(\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_{\nu-1} 1][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]} \right) (\lambda_1 \ldots \lambda_{\nu-1} 1) \\ & + \sum_{\lambda_1 \ldots \lambda_\nu \geqslant 2} \left(\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_\nu][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]} \right. \\ & \left. - \sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_\nu 2][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]} \right) (\lambda_1 \ldots \lambda_\nu), \end{aligned} \tag{2.4}$$

where $g_{\rho\sigma\tau}$ is the multiplicity of the irrep τ of $\mathrm{U}(n)$ in the product representation $\rho \times \sigma$, and $h_1^s, \ldots, h_{\nu+1}^s$ are even integers. In equation (2.3), the prime (double prime) over the summation symbol over $\lambda_1, \ldots, \lambda_\nu$ means that these numbers are restricted to those values such that $h_1 + \ldots + h_{\nu+1} - (\lambda_1 + \ldots + \lambda_\nu)$ is even (odd).

The modified branching rule differs in two respects from the standard one. Firstly it contains some additional positive terms, respectively the second and the third ones in equations (2.3) and (2.4). Next it contains some negative terms which compensate part of the terms coming from the ordinary branching rule. Such is the last term in equation (2.4), while none of this kind is present in equation (2.3). It turns out that the only case where the modified branching rule does not contain any negative term is where $d = \nu + 1$ and $n = 2\nu + 1$, consequently easing the solution of the state labelling problem for this case with respect to the remaining ones. In § 4, we shall therefore start with this example. Then we shall consider in § 5 the case where $d = \nu + 1$ and $n = 2\nu$, corresponding to equation (2.4). All the difficulties of the general case already being present, it will be sufficient to illustrate the general solution of the state labelling problem.

Let us now determine the number of missing labels distinguishing the equivalent irreps of $\mathrm{O}(n)$ $[\mathrm{U}(d)]$ as contained in a given irrep of $\mathrm{U}(n)$ $[\mathrm{Sp}(2d, R)]$ when $d > \nu$. As seen in equations (2.1) and (2.2), neither the irrep of $\mathrm{U}(n)$ nor that of $\mathrm{Sp}(2d, R)$ are the most general irreps we can have for such groups. This renders Racah's formula (1965) for the number of missing labels useless, hence we have to count them by using appropriate chains of subgroups.

To characterise the row of the $\mathrm{U}(n)$ irrep $[h_1 \ldots h_d]$, we could use the canonical chain of subgroups (Gel'fand and Tseitlin 1950)

$$\begin{array}{ccccccccccccc} \mathrm{U}(n-1) & \supset & \mathrm{U}(n-2) & \supset \ldots \supset & \mathrm{U}(d) & \supset & \mathrm{U}(d-1) & \supset \ldots \supset & \mathrm{U}(2) & \supset & \mathrm{U}(1). \\ d & & d & & d & & d-1 & & 2 & & 1 \end{array} \tag{2.5}$$

The number of labels specifying the irreps is listed below each corresponding group. Altogether they would supply $dn - \frac{1}{2}d(d+1)$ quantum numbers. However, instead of chain (2.5), we actually make use of the $\mathrm{O}(n)$ group and its canonical chain of subgroups (Gel'fand and Tseitlin 1950), i.e.,

$$\begin{array}{ccccccccccccc} \mathrm{O}(2\nu+1) & \supset & \mathrm{O}(2\nu) & \supset & \mathrm{O}(2\nu-1) & \supset & \mathrm{O}(2\nu-2) & \supset \ldots \supset & \mathrm{O}(3) & \supset & \mathrm{O}(2) \\ \nu & & \nu & & \nu-1 & & \nu-1 & & 1 & & 1 \end{array}$$

$$\text{when } n = 2\nu + 1, \tag{2.6a}$$

and

$$\begin{array}{ccccccccccc} \mathrm{O}(2\nu) & \supset & \mathrm{O}(2\nu-1) & \supset & \mathrm{O}(2\nu-2) & \supset \ldots \supset & \mathrm{O}(3) & \supset & \mathrm{O}(2) \\ \nu & & \nu-1 & & \nu-1 & & 1 & & 1 \end{array} \qquad \text{when } n = 2\nu. \tag{2.6b}$$

They supply respectively $\nu(\nu+1)$ and ν^2 quantum numbers. For $d = \nu + a$, where a is any positive integer, the number of missing labels is therefore equal to

$$\tfrac{1}{2}d(d-1) - a(a-1) \qquad \text{when } n = 2\nu + 1, \tag{2.7a}$$

and

$$\tfrac{1}{2}d(d-1) - a^2 \qquad \text{when } n = 2\nu. \tag{2.7b}$$

Let us recall that for $d \leqslant \nu$, the number of missing labels is $\frac{1}{2}d(d-1)$, irrespective of the difference $\nu - d$. When going from $d \leqslant \nu$ to $d > \nu$, it therefore decreases, except for the case $d=\nu+1$, $n=2\nu+1$, where it remains equal to $\frac{1}{2}d(d-1)$. This property must be compared to the presence or the absence of negative terms in Littlewood's modified branching rule.

3. Highest weight states of equivalent O(n) irreps belonging to a given U(n) irrep

In this section, we wish to address ourselves to the construction of the HWS P of equivalent $\mathrm{O}(n)$ irreps $(\lambda_1 \ldots \lambda_\nu)$, belonging to a given $\mathrm{U}(n)$ irrep $[h_1 \ldots h_d]$. As a consequence of the complementarity relationship mentioned in § 2, they are simultaneously the HWS of equivalent $\mathrm{U}(d)$ irreps $[h_1 \ldots h_d]$, contained in the $\mathrm{Sp}(2d, R)$ irrep $\langle(\frac{1}{2}n)^{d-\nu}, \lambda_\nu+\frac{1}{2}n, \ldots, \lambda_1+\frac{1}{2}n\rangle$. We shall specifically be more concerned with the changes to be made in the construction detailed in I when d becomes larger than ν.

In the Bargmann representation (1961), the HWS P is represented by a polynomial $P(z_{is})$ in dn complex variables z_{is}, $i=1,\ldots,d$, $s=1,\ldots,n$,

$$
\begin{aligned}
P(z_{is}) &= \left\langle z_{is} \left| \begin{matrix} \langle(\frac{1}{2}n)^{d-\nu}, \lambda_\nu+n/2, \ldots, \lambda_1+n/2\rangle & [h_1 \ldots h_d] \\ (\Gamma^s)[h_1 \ldots h_d] & ;(\Gamma^s)(\lambda_1 \ldots \lambda_\nu) \\ \max & \max \end{matrix} \right.\right\rangle \\
&= \langle z_{is} | [h_1 \ldots h_d] \max; (\lambda_1 \ldots \lambda_\nu) \max; (\Gamma^s)\rangle,
\end{aligned}
\tag{3.1}
$$

where (Γ^s) denotes the whole set of missing labels. Each $P(z_{is})$ is a solution of the following system of equations

$$
H_\alpha P = \lambda_\alpha P, \qquad A^\beta_\alpha P = 0, \qquad \beta < \alpha, \tag{3.2a, b}
$$

$$
D^\beta_\alpha P = 0, \qquad \beta < \alpha, \qquad E^\alpha_n P = 0 \qquad (\text{only when } n = 2\nu+1), \tag{3.2c, d}
$$

$$
C_{ii} P = h_i P, \qquad C_{ij} P = 0, \qquad i < j, \tag{3.2e, f}
$$

where $\alpha, \beta = 1, \ldots, \nu$, and $i, j = 1, \ldots, d$. The operators H_α, and $A^\beta_\alpha (\beta < \alpha)$, $D^\beta_\alpha (\beta < \alpha)$, E^α_n (only when $n = 2\nu+1$) are respectively the weight and raising generators of $\mathrm{O}(n)$, while the operators C_{ii} and $C_{ij} (i<j)$ play the same role for $\mathrm{U}(d)$. Their definition in terms of z_{is} and $\partial/\partial z_{is}$ was given in I and will not be repeated here.

We start with the replacement of the dn variables z_{is} by the variables $a_{i\alpha}$, $b_{i\alpha}$, and c_i (the latter only for $n=2\nu+1$), $i=1,\ldots,d$, $\alpha=1,\ldots,\nu$, characterised by a definite weight with respect to $\mathrm{O}(n)$, and defined in equation (I3.1). In terms of these new variables, the weight and raising generators of $\mathrm{O}(n)$ can be written as

$$
\begin{aligned}
H_\alpha &= \sum_i \left(a_{i\alpha} \frac{\partial}{\partial a_{i\alpha}} - b_{i\alpha} \frac{\partial}{\partial b_{i\alpha}} \right), & A^\beta_\alpha &= \sum_i \left(a_{i\alpha} \frac{\partial}{\partial b_{i\beta}} - a_{i\beta} \frac{\partial}{\partial b_{i\alpha}} \right), \\
D^\beta_\alpha &= \sum_i \left(a_{i\beta} \frac{\partial}{\partial a_{i\alpha}} - b_{i\alpha} \frac{\partial}{\partial b_{i\beta}} \right), & E^\alpha_n &= \sum_i \left(c_i \frac{\partial}{\partial b_{i\alpha}} - a_{i\alpha} \frac{\partial}{\partial c_i} \right),
\end{aligned}
\tag{3.3}
$$

while the $\mathrm{U}(d)$ generators become

$$
\begin{aligned}
C_{ij} &= \sum_\alpha \left(a_{i\alpha} \frac{\partial}{\partial a_{j\alpha}} + b_{i\alpha} \frac{\partial}{\partial b_{j\alpha}} \right) & &\text{if } n = 2\nu, \\
&= \sum_\alpha \left(a_{i\alpha} \frac{\partial}{\partial a_{j\alpha}} + b_{i\alpha} \frac{\partial}{\partial b_{j\alpha}} \right) + c_i \frac{\partial}{\partial c_j} & &\text{if } n = 2\nu+1.
\end{aligned}
\tag{3.4}
$$

Proceeding as in I, we next try to introduce the $\frac{1}{2}d(d+1)$ O(n) scalars $w_{ij} = w_{ji}$, defined in equation (I3.2), by eliminating the same number of $b_{i\alpha}$-type variables. However it turns out that when d becomes larger than ν, with the exception of $d = \nu+1$ and $n = 2\nu+1$, such a transformation is no more possible, because the resulting set of variables would not be functionally independent. To obtain a set of independent variables when $d = \nu + a$ and a is any positive integer, we have to eliminate $\frac{1}{2}a(a-1)$ or $\frac{1}{2}a(a+1)$ $a_{i\alpha}$-type variables, but only $\frac{1}{2}[d(d+1)-a(a-1)]$ or $\frac{1}{2}[d(d+1)-a(a+1)]$ $b_{i\alpha}$-type variables, according as $n = 2\nu+1$ or $n = 2\nu$. The new variables are defined by

$$\begin{aligned}
u_{i\sigma} &= a_{i\sigma}, && \sigma = 1, \ldots, \min(\nu, n-i), \\
v_{i\rho} &= b_{i\rho}, && \rho = 1, \ldots, \nu - i, \\
&= c_i, && \rho = \nu - i + 1 \text{ (only when } n = 2\nu+1), \\
w_{ij} &= \sum_{\alpha=1}^{\nu} (a_{i\alpha} b_{j\alpha} + a_{j\alpha} b_{i\alpha}) && \text{when } n = 2\nu, \\
&= \sum_{\alpha=1}^{\nu} (a_{i\alpha} b_{j\alpha} + a_{j\alpha} b_{i\alpha}) + c_i c_j && \text{when } n = 2\nu+1,
\end{aligned} \tag{3.5}$$

where i and j run from 1 to d, appreciating that in $v_{i\rho}$, i does not go beyond $\nu-1$ or ν whenever $n = 2\nu$ or $n = 2\nu+1$.

The polynomials $P(z_{is})$ become $P(u_{i\sigma}, v_{i\rho}, w_{ij})$ functions, which may be non-analytic since the change of variables (3.5) is not linear. For the time being let us restrict ourselves to analytic functions. Equations (3.2b) and (3.2d) can be solved in the same way as in I and impose that $P(u_{i\sigma}, v_{i\rho}, w_{ij})$ depends only upon the $u_{i\sigma}$ and w_{ij} variables. On the contrary, equations (3.2a, c, e, f) are substantially modified with respect to I. The most dramatic change occurs in the U(d) generators, where a separation, similar to equation (I4.1), into two terms depending only upon the $u_{i\sigma}$ or the w_{ij} variables is no more valid, except for $d = \nu+1$, $n = 2\nu+1$.

We therefore conclude that the procedure used in I to construct $P(u_{i\sigma}, w_{ij})$ is not directly applicable here except for the case $d = \nu+1$, $n = 2\nu+1$, which is going to be reviewed in the next section. The construction of $P(u_{i\sigma}, w_{ij})$ is postponed until § 5 when the $a_{i\alpha}$ variables are not all functionally independent of the w_{ij} ones.

4. Solution of the state labelling problem for $(\nu+1)$-row irreps of U$(2\nu+1)$

When $d = \nu+1$ and $n = 2\nu+1$, no $a_{i\alpha}$-type variable has to be eliminated, and equation (3.5) reduces to equation (I3.3). The conclusions of I may therefore be extended to the present case.

Let us consider the whole set of HWS $P(u_{i\alpha}, w_{ij})$ of equivalent O$(2\nu+1)$ [or SO$(2\nu+1)$] irreps $(\lambda_1 \ldots \lambda_\nu)$, belonging to the U$(2\nu+1)$ [or SU$(2\nu+1)$] irrep $[h_1 \ldots h_{\nu+1}]$. According to I, those members of the set which are analytic functions in $u_{i\alpha}$ and w_{ij} can be written as

$$\begin{aligned}
&\langle u_{i\alpha}, w_{ij} | [h_1 \ldots h_{\nu+1}] \max; (\lambda_1 \ldots \lambda_\nu) \max; (\Gamma^s) \rangle \\
&\quad = \sum_{(h^s)(\lambda)} \langle [\lambda_1 \ldots \lambda_\nu](\lambda), [h_1^s \ldots h_{\nu+1}^s](h^s) | [h_1 \ldots h_{\nu+1}] \max; (\gamma^s) \rangle \\
&\qquad \times \langle u_{i\alpha} | [\lambda_1 \ldots \lambda_\nu](\lambda); (\lambda_1 \ldots \lambda_\nu) \max \rangle \langle w_{ij} | [h_1^s \ldots h_{\nu+1}^s](h^s); (0) \max \rangle.
\end{aligned} \tag{4.1}$$

In equation (4.1), $h_1^s, \ldots, h_{\nu+1}^s$ are some even integers, (h^s) and (λ) are Gel'fand patterns (Gel'fand and Tseitlin 1950), (γ^s) is an operator pattern (Biedenharn *et al* 1967), (Γ^s) is defined by

$$(\Gamma^s) = \begin{pmatrix} (\gamma^s) \\ [h_1^s \ldots h_{\nu+1}^s] \end{pmatrix}, \tag{4.2}$$

and the first factor on the right-hand side is a shorthand notation for a $\mathrm{U}(\nu+1)$ Wigner coefficient

$$\langle [\lambda_1 \ldots \lambda_\nu](\lambda), [h_1^s \ldots h_{\nu+1}^s](h^s) | [h_1 \ldots h_{\nu+1}] \max; (\gamma^s) \rangle$$
$$= \left\langle \begin{matrix} [h_1 \ldots h_{\nu+1}] \\ \max \end{matrix} \middle| \begin{pmatrix} (\gamma^s) \\ [h_1^s \ldots h_{\nu+1}^s] \\ (h^s) \end{pmatrix} \middle| \begin{matrix} [\lambda_1 \ldots \lambda_\nu] \\ (\lambda) \end{matrix} \right\rangle. \tag{4.3}$$

(Γ^s) provides the $\frac{1}{2}d(d-1)$ missing labels necessary to completely specify the HWS.

Since $h_1^s, \ldots, h_{\nu+1}^s$ are even, only HWS of irreps $(\lambda_1 \ldots \lambda_\nu)$ such that $h_1 + \ldots + h_{\nu+1} - (\lambda_1 + \ldots + \lambda_\nu)$ is even can be obtained in this way, and the number of such independent states is equal to

$$\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_\nu][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]}. \tag{4.4}$$

Therefore they only account for the first term of Littlewood's modified branching rule, given in equation (2.3). The remaining HWS, corresponding to the second term of equation (2.3) for which $h_1 + \ldots + h_{\nu+1} - (\lambda_1 + \ldots + \lambda_\nu)$ is odd, must be non-analytic functions in $u_{i\alpha}$ and w_{ij}.

When considering increasing values of $h_1, \ldots, h_{\nu+1}$, a non-analytic HWS appears for the first time for the irreps $[1^{\nu+1}]$ of $\mathrm{U}(2\nu+1)$ and (1^ν) of $\mathrm{O}(2\nu+1)$ with a multiplicity one corresponding to $h_1^s = \ldots = h_{\nu+1}^s = 0$. In terms of the variables $a_{i\alpha}$, $b_{i\alpha}$, and c_i, it is however a polynomial whose explicit form can be easily written down by solving equations (3.2), (3.3), and (3.4) for $h_1 = \ldots = h_{\nu+1} = \lambda_1 = \ldots = \lambda_\nu = 1$. It is given by

$$\langle a_{i\alpha}, b_{i\alpha}, c_i | [1^{\nu+1}] \max; (1^\nu) \max \rangle = \sum_i (-1)^{\nu+1-i} a_{1 \ldots i-1\, i+1 \ldots \nu+1, 1 \ldots \nu} c_i, \tag{4.5}$$

were $a_{1 \ldots i-1\, i+1 \ldots \nu+1, 1 \ldots \nu}$ is the determinant of the variables $a_{j\alpha}$, where $j = 1, \ldots, i-1, i+1, \ldots, \nu+1$, and $\alpha = 1, \ldots, \nu$. One can easily check that the HWS (4.5) becomes a non-analytic function once written in terms of the variables $u_{i\alpha}$, w_{ij}. This property results from the following identity

$$\sum_{ij} (-1)^{i+j} u_{1 \ldots i-1\, i+1 \ldots \nu+1, 1 \ldots \nu} u_{1 \ldots j-1\, j+1 \ldots \nu+1, 1 \ldots \nu} w_{ij}$$
$$= \left[\sum_i (-1)^{\nu+1-i} a_{1 \ldots i-1\, i+1 \ldots \nu+1, 1 \ldots \nu} c_i \right]^2, \tag{4.6}$$

which is proved by applying equation (3.5) to the left-hand side and using some determinant standard properties.

We now assert that all the HWS, corresponding to irreps $[h_1 \ldots h_{\nu+1}]$ and $(\lambda_1 \ldots \lambda_\nu)$ for which $h_1 + \ldots + h_{\nu+1} - (\lambda_1 + \ldots + \lambda_\nu)$ is odd, can be built from the HWS of the irreps $[1^{\nu+1}]$ and (1^ν), given in equation (4.5), and some HWS which are analytic functions in $u_{i\alpha}$ and w_{ij}. More explicitly they can be written as

$$\langle a_{i\alpha}, b_{i\alpha}, c_i | [1^{\nu+1}] \max; (1^\nu) \max\rangle$$
$$\times \langle u_{i\alpha}, w_{ij} | [h_1 - 1, \ldots, h_{\nu+1} - 1] \max; (\lambda_1 - 1, \ldots, \lambda_\nu - 1) \max; (\Gamma^s)\rangle, \tag{4.7}$$

where the latter factor is given by an equation similar to equation (4.1), and (Γ^s) retains the same meaning as before.

To prove this assertion, we note that (i) the function defined in equation (4.7) is a solution of equation (3.2); (ii) since $(h_1 - 1) + \ldots + (h_{\nu+1}) - [(\lambda_1 - 1) + \ldots + (\lambda_\nu - 1)]$ is even, the second factor in equation (4.7) is an analytic function in $u_{i\alpha}$, w_{ij}, whose explicit form can be obtained from equation (4.1); (iii) the number of independent functions, obtained by considering all possible (Γ^s) in equation (4.7), is equal to

$$\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 - 1, \ldots, \lambda_\nu - 1][h_1^s \ldots h_{\nu+1}^s][h_1 - 1, \ldots, h_{\nu+1} - 1]} = \sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1 \ldots \lambda_\nu 1][h_1^s \ldots h_{\nu+1}^s][h_1 \ldots h_{\nu+1}]}, \tag{4.8}$$

and accounts for the second term of the modified branching rule, given in equation (2.3), thus completing the proof.

The state labelling problem for the $(\nu+1)$-row irreps of $U(2\nu+1)$, when reduced with respect to $O(2\nu+1)$, is therefore solved. We shall turn now to the more difficult problem of the $(\nu+1)$-row irreps of $U(2\nu)$.

5. Solution of the state labelling problem for $(\nu+1)$-row irreps of $U(2\nu)$

When $d = \nu + 1$ and $n = 2\nu$, by inverting equation (3.5) one of the $a_{i\alpha}$ variables, namely $a_{\nu+1,\nu}$, can be expressed as a function of $u_{i\sigma}$, $v_{i\rho}$, and w_{ij} (it will turn out that it only depends upon $u_{i\sigma}$ and w_{ij}). For such a purpose, let us eliminate the $b_{i\alpha}$ variables between the set of $\frac{1}{2}(\nu+1)(\nu+2)$ equations

$$w_{ij} = \sum_\alpha (a_{i\alpha} b_{j\alpha} + a_{j\alpha} b_{i\alpha}), \qquad i \leq j = 1, \ldots, \nu + 1. \tag{5.1}$$

This is accomplished by multiplying both sides of equation (5.1) by $(-1)^{i+j} \times a_{1 \ldots i-1 i+1 \ldots \nu+1, 1 \ldots \nu} a_{1 \ldots j-1 j+1 \ldots \nu+1, 1 \ldots \nu}$, and summing over i and j from 1 to $\nu+1$. The result can be written as

$$\sum_{ij} (-1)^{i+j} a_{1 \ldots i-1 i+1 \ldots \nu+1, 1 \ldots \nu} a_{1 \ldots j-1 j+1 \ldots \nu+1, 1 \ldots \nu} w_{ij} = 0, \tag{5.2a}$$

because the sum over α on the right-hand side contains a factor

$$\sum_i (-1)^i a_{i\alpha} a_{1 \ldots i-1 i+1 \ldots \nu+1, 1 \ldots \nu} = -a_{1 \ldots \nu+1, \alpha 1 \ldots \nu}, \tag{5.3}$$

which is equal to zero for any $\alpha = 1, \ldots, \nu$. When replacing $a_{i\sigma}$ by $u_{i\sigma}$ for $i, \sigma = 1, \ldots, \nu$, and $i = \nu + 1$, $\sigma = 1, \ldots, \nu - 1$, in equation (5.2$a$), we obtain a second degree equation for $a_{\nu+1,\nu}$, whose coefficients are polynomials in $u_{i\sigma}$ and w_{ij}. Its solution leads to the sought for expression of $a_{\nu+1,\nu}$ in terms of $u_{i\sigma}$ and w_{ij}. We do not write the latter here since its explicit form will not be needed. Instead we shall be interested in the interpretation of equation (5.2a), with whose discussion we shall now proceed.

Should n be greater than or equal to $2d=2\nu+2$, instead of being equal to 2ν as it is here, we should remain within the case treated in I. Then all the $a_{i\alpha}$ variables, including $a_{\nu+1,\nu}$, would be independent of the w_{ij} variables and equal to $u_{i\alpha}$. In this case, the left-hand side of equation (5.2a) would become

$$\sum_{ij}(-1)^{i+j}u_{1\ldots i-1\, i+1\ldots \nu+1,1\ldots\nu}u_{1\ldots j-1\, j+1\ldots\nu+1,1\ldots\nu}w_{ij}, \tag{5.4}$$

and would of course be different from zero. By introducing equation (5.4) into equation (3.2), it is straightforward to show that this polynomial in $u_{i\alpha}$ and w_{ij} would just be the HWS of the irrep (2^ν) of O(n) belonging to the irrep $[2^{\nu+1}]$ of U(n), i.e., $\langle u_{i\alpha}, w_{ij}|[2^{\nu+1}]\max; (2^\nu)\max\rangle$, where no extra label (Γ^s) is needed. Returning now to the case $n=2\nu$, i.e., replacing $u_{i\alpha}$ by $a_{i\alpha}$, we conclude that equation (5.2a) is equivalent to the condition

$$\langle a_{i\alpha}, w_{ij}|[2^{\nu+1}]\max; (2^\nu)\max\rangle = 0. \tag{5.2b}$$

The latter agrees with Littlewood's modified branching rule, given in equation (2.4), showing that the irrep (2^ν) is not contained in $[2^{\nu+1}]$ when $n=2\nu$.

The equivalence of equations (5.2a) and (5.2b) suggests a convenient procedure to take into account the functional dependence of $a_{i\alpha}$ and w_{ij} when $n=2\nu$: starting with the case $n\geqslant 2d=2\nu+2$, use the results of I to write the HWS in a form similar to equation (4.1), with $u_{i\alpha}$ replaced by $a_{i\alpha}$; then pass to the case $n=2\nu$ by directly imposing condition (5.2b) on the HWS.

To implement this programme, we note that when $n\geqslant 2d=2\nu+2$, $h_{\nu+1}\geqslant 2$, and $\lambda_\nu\geqslant 2$, the products

$$\langle a_{i\alpha}, w_{ij}|[2^{\nu+1}]\max; (2^\nu)\max\rangle \\ \times\langle a_{i\alpha}, w_{ij}|[h_1-2,\ldots,h_{\nu+1}-2]\max; (\lambda_1-2,\ldots,\lambda_\nu-2)\max; (\Gamma^s)\rangle \tag{5.5a}$$

are of highest weight, respectively equal to $h_1,\ldots,h_{\nu+1}$, and $\lambda_1,\ldots,\lambda_\nu$ with respect to U($\nu+1$) and O(n). They must therefore be linear combinations of the HWS of the equivalent O(n) irreps $(\lambda_1\ldots\lambda_\nu)$ belonging to the irrep $[h_1\ldots h_{\nu+1}]$ of U(n),

$$\sum_{(\Gamma^{s'})} A^{(\Gamma^s)}_{(\Gamma^{s'})}([h_1\ldots h_{\nu+1}](\lambda_1\ldots\lambda_\nu))\langle a_{i\alpha}, w_{ij}|[h_1\ldots h_{\nu+1}]\max; (\lambda_1\ldots\lambda_\nu)\max; (\Gamma^{s'})\rangle. \tag{5.5b}$$

In the appendix, we show that the numerical coefficients $A^{(\Gamma^s)}_{(\Gamma^{s'})}([h_1\ldots h_{\nu+1}](\lambda_1\ldots\lambda_\nu))$ of these linear combinations are just recoupling coefficients of four U($\nu+1$) irreps, which we represent by a notation similar to that commonly used for SU(2) (Edmonds 1957):

$$\begin{aligned} A^{(\Gamma^s)}_{(\Gamma^{s'})}&([h_1\ldots h_{\nu+1}](\lambda_1\ldots\lambda_\nu)) \\ &=\langle(([\lambda_1-2,\ldots,\lambda_\nu-2][2^\nu])[\lambda_1\ldots\lambda_\nu], ([h_1^s\ldots h_{\nu+1}^s][2])[h_1^{s'}\ldots h_{\nu+1}^{s'}], \\ &\quad(\gamma^{s'})[h_1\ldots h_{\nu+1}]|([\lambda_1-2,\ldots,\lambda_\nu-2][h_1^s\ldots h_{\nu+1}^s]) \\ &\quad(\gamma^s)[h_1-2,\ldots,h_{\nu+1}-2], ([2^\nu][2])[2^{\nu+1}], [h_1\ldots h_{\nu+1}]\rangle. \end{aligned} \tag{5.6}$$

In equation (5.6), (Γ^s) and $(\Gamma^{s'})$ are respectively equal to the right-hand side of equation (4.2) and to a similar expression containing primed symbols.

Therefore we are, at least in principle, able to write explicitly the relations obtained by equating equations (5.5a) and (5.5b). For given irreps $[h_1\ldots h_{\nu+1}]$ and $(\lambda_1\ldots\lambda_\nu)$

such that $h_{\nu+1}$ and $\lambda_\nu \geqslant 2$, they are all independent of one another, and their number, equal to the number of operator patterns (Γ^s), is the multiplicity of $(\lambda_1-2, \ldots, \lambda_\nu-2)$ in $[h_1-2, \ldots, h_{\nu+1}-2]$, or in accordance with Littlewood's branching rule

$$\sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1-2,\ldots,\lambda_\nu-2][h_1^s\ldots h_{\nu+1}^s][h_1-2,\ldots,h_{\nu+1}-2]} = \sum_{h_1^s \ldots h_{\nu+1}^s} g_{[\lambda_1\ldots\lambda_\nu 2][h_1^s\ldots h_{\nu+1}^s][h_1\ldots h_{\nu+1}]}. \tag{5.7}$$

When we pass to the $n=2\nu$ case and take equation (5.2b) into account, it is clear that the linear combinations (5.5b) must be equal to zero since the expressions (5.5a) vanish. Consequently, when $h_{\nu+1}$ and $\lambda_\nu \geqslant 2$, the HWS (4.1) are no longer independent but related by the set of equations

$$\sum_{(\Gamma^{s'})} A^{(\Gamma^s)}_{(\Gamma^{s'})}([h_1 \ldots h_{\nu+1}](\lambda_1 \ldots \lambda_\nu)) \times \langle a_{i\alpha}, w_{ij} | [h_1 \ldots h_{\nu+1}] \max; (\lambda_1 \ldots \lambda_\nu) \max; (\Gamma^{s'}) \rangle = 0, \tag{5.8}$$

whose number is given by equation (5.7).

A solution of the $U(2\nu) \supset O(2\nu)$ state labelling problem, for the irreps $[h_1 \ldots h_{\nu+1}]$ and $(\lambda_1 \ldots \lambda_\nu)$ with $h_{\nu+1}$ and $\lambda_\nu \geqslant 2$, may therefore be obtained in the following way. Among the HWS (4.1), we only retain a set of independent states by taking equation (5.8) into account. The number of such independent states is the difference between the numbers defined in equations (4.4) and (5.7), and agrees with the two last terms of Littlewood's modified branching rule (2.4). The patterns (Γ^s) can still be used to specify the independent HWS. However they provide us with a redundant characterisation, since they contain $\frac{1}{2}d(d-1)$ independent labels and, according to equation (2.7b), we only need $\frac{1}{2}d(d-1)-1$ additional quantum numbers.

Let us now turn to the cases where $\lambda_\nu \geqslant 2$ and $h_{\nu+1} < 2$, or $\lambda_\nu < 2$ and $h_{\nu+1}$ is arbitrary. It is clear from equation (5.2b) that they are only slightly affected by the functional dependence of the $a_{i\alpha}$ and w_{ij} variables. The states (4.1), with $u_{i\alpha}$ replaced by $a_{i\alpha}$, remain independent HWS. The functional dependence of the $a_{i\alpha}$ and w_{ij} variables only appears when we write the states in terms of $u_{i\sigma}$ and w_{ij} because $a_{\nu+1,\nu}$ is a complicated function of these variables. In doing so, the polynomials in $a_{i\alpha}$, w_{ij} are converted into non-analytic functions in $u_{i\sigma}$, w_{ij}. Such HWS account for the first, the second, and the fourth terms of equation (2.4), as well as the fifth one whenever $h_{\nu+1} < 2$.

For $\lambda_{\nu-1} \geqslant 1$, $\lambda_\nu = 0$, and $h_{\nu+1} \geqslant 1$, it remains for the HWS corresponding to the third term of equation (2.4) to be constructed. They are non-analytic functions in $a_{i\alpha}$ and w_{ij}, whose explicit form can be found by an analysis similar to that used in § 4 to construct the states (4.7). They can be written as

$$\langle a_{i\alpha}, b_{i\alpha} | [1^{\nu+1}] \max; (1^{\nu-1}) \max \rangle \times \langle u_{i\sigma}, w_{ij} | [h_1-1, \ldots, h_{\nu+1}-1] \max; (\lambda_1-1, \ldots, \lambda_{\nu-1}-1) \max; (\Gamma^s) \rangle,$$

where (5.9)

$$\langle a_{i\alpha}, b_{i\alpha} | [1^{\nu+1}] \max; (1^{\nu-1}) \max \rangle = \sum_i (-1)^{\nu+1-i} a_{1\ldots i-1\, i+1 \ldots \nu+1, 1 \ldots \nu} b_{i\nu}, \tag{5.10}$$

and

$$\sum_{\substack{i<j \\ k<l}} [(-1)^{i+j+k+l+1} u_{1\ldots i-1\, i+1 \ldots j-1\, j+1 \ldots \nu+1, 1 \ldots \nu} u_{1 \ldots k-1\, k+1 \ldots l-1\, l+1 \ldots \nu+1, 1 \ldots \nu} w_{ij,kl}] = \left[\sum_i (-1)^{\nu+1-i} a_{1\ldots i-1\, i+1 \ldots \nu+1, 1\ldots\nu} b_{i\nu}\right]^2. \tag{5.11}$$

The second factor in equation (5.9) is a polynomial in $a_{i\alpha}$, w_{ij} of type (4.1). Since $\lambda_\nu = 0$, it does not depend upon $a_{\nu+1,\nu}$ and is therefore also a polynomial in $u_{i\sigma}$ and w_{ij}. The number of independent functions obtained by considering all possible (Γ^s) in equation (5.9) is equal to

$$\sum_{h_1^s \dots h_{\nu+1}^s} g_{[\lambda_1-1,\dots,\lambda_{\nu-1}-1][h_1^s \dots h_{\nu+1}^s][h_1-1,\dots,h_{\nu+1}-1]}$$
$$= \sum_{h_1^s \dots h_{\nu+1}^s} g_{[\lambda_1 \dots \lambda_{\nu-1} 1^2][h_1^s \dots h_{\nu+1}^s][h_1 \dots h_{\nu+1}]}, \tag{5.12}$$

and accounts for the third term of equation (2.4).

We have therefore completed the solution of the state labelling problem for $(\nu+1)$-row irreps of $U(2\nu)$, when reduced with respect to $O(2\nu)$.

6. Conclusion

In the two previous sections, we solved in detail the state labelling problem respectively for $(\nu+1)$-row irreps of $U(2\nu+1)$ and $U(2\nu)$. In this concluding section, we would like to outline the procedure for generalising our results to d-row irreps of $U(n)$, when $d = \nu + a$ and a is some integer larger than one.

As mentioned in § 2, Newell's modification rules introduce two types of additional terms into Littlewood's branching rule, namely positive and negative ones. We shall discuss them separately.

The existence of negative terms in Littlewood's modified branching rule is related to the functional dependence of the $a_{i\alpha}$ and w_{ij} variables, and the drop in the number of missing labels that both take place when d exceeds ν (or $\nu+1$ for $n = 2\nu+1$). Whenever n is even or odd, $\frac{1}{2}a(a+1)$ or $\frac{1}{2}a(a-1)$ $a_{i\alpha}$-type variables can be written in terms of the w_{ij} variables and the remaining $a_{i\alpha}$ ones. The corresponding $\frac{1}{2}a(a+1)$ or $\frac{1}{2}a(a-1)$ equations express the fact that some HWS corresponding to low values of $h_1, \dots, h_d$, and $\lambda_1, \dots, \lambda_\nu$, which were present for $n \geq 2d$, disappear for $n < 2d$ and must therefore be cancelled. As a consequence, some of the HWS constructed in I are no more independent. The relations they satisfy can be deduced from the above mentioned equations by a procedure similar to that devised in the previous section, and they involve appropriate recoupling coefficients of $U(d)$. Their number is equal to the absolute value of the negative terms in Littlewood's modified branching rule. The independent HWS that are left after eliminating the dependent ones with the help of such relations, can still be characterised by (Γ^s) patterns, but the latter provides us with too many a^2 or $a(a-1)$ labels whenever n is even or odd.

The existence of additional positive terms in Littlewood's modified branching rule means that some HWS of $O(n)$ irreps are non-analytic functions in $a_{i\alpha}$ and w_{ij}. Fortunately this non-analyticity arises from some HWS corresponding to low values of $h_1, \dots, h_d$, and $\lambda_1, \dots, \lambda_\nu$. The latter can be easily constructed as polynomials in $a_{i\alpha}$, $b_{i\alpha}$, and c_i. All the non-analytic HWS can then be obtained by combining them with analytic ones.

From the above discussion, we conclude that the solution of the state labelling problem for d-row irreps of $U(n)$, when reduced with respect to $O(n)$, formulated in I for $d \leq [\frac{1}{2}n]$, can in principle be extended to all d values (although when d is large with respect to $[\frac{1}{2}n]$, it would become extremely complicated). For $d > [\frac{1}{2}n]$, it is directly connected with Littlewood's modified branching rule.

The solution proposed in both papers (I and present) will of course remain formal in most cases, because the numerical values of the necessary Wigner and recoupling coefficients of $U(d)$ are unknown. In the remaining cases however, detailed expressions of the HWS can be written down as will be shown in a forthcoming paper of the present series, due to deal with the $SU(3)\supset SO(3)$ group chain.

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Appendix. Proof of equation (5.6)

In this appendix, we would like to show that the coefficients $A^{(\Gamma^s)}_{(\Gamma^{s'})}([h_1\ldots h_{\nu+1}]$ $(\lambda_1\ldots\lambda_\nu))$, appearing on the right-hand side of the following equation

$$
\begin{aligned}
&\langle a_{i\alpha}, w_{ij}|[2^{\nu+1}]\max;(2^\nu)\max\rangle\\
&\qquad\times\langle a_{i\alpha}, w_{ij}|[h_1-2,\ldots,h_{\nu+1}-2]\max;(\lambda_1-2,\ldots,\lambda_\nu-2)\max;(\Gamma^s)\rangle\\
&\quad=\sum_{(\Gamma^{s'})}A^{(\Gamma^s)}_{(\Gamma^{s'})}([h_1\ldots h_{\nu+1}](\lambda_1\ldots\lambda_\nu))\\
&\qquad\times\langle a_{i\alpha}, w_{ij}|[h_1\ldots h_{\nu+1}]\max;(\lambda_1\ldots\lambda_\nu)\max;(\Gamma^{s'})\rangle
\end{aligned}\tag{A1}
$$

are equal to the recoupling coefficients of $U(\nu+1)$, in accordance with equation (5.6).

For such purposes, let us expand each of the HWS in equation (A1) into products of HWS depending only upon $a_{i\alpha}$ or w_{ij} by applying equation (4.1). Equation (A1) becomes

$$
\begin{aligned}
&\sum_{(h_0^s)(\lambda_0)(h^s)(\lambda)}(\langle[2^\nu](\lambda_0),[2](h_0^s)|[2^{\nu+1}]\max;(\min)\rangle\\
&\qquad\times\langle[\lambda_1-2,\ldots,\lambda_\nu-2](\lambda),[h_1^s\ldots h_{\nu+1}^s](h^s)\\
&\qquad\times|[h_1-2,\ldots,h_{\nu+1}-2]\max;(\gamma^s)\rangle\\
&\qquad\times\langle a_{i\alpha}|[2^\nu](\lambda_0);(2^\nu)\max\rangle\\
&\qquad\times\langle a_{i\alpha}|[\lambda_1-2,\ldots,\lambda_\nu-2](\lambda);(\lambda_1-2,\ldots,\lambda_\nu-2)\max\rangle\\
&\qquad\times\langle w_{ij}|[2](h_0^s);(0)\max\rangle\langle w_{ij}|[h_1^s\ldots h_{\nu+1}^s](h^s);(0)\max\rangle)\\
&\quad=\sum_{(\Gamma^{s'})}\bigg[A^{(\Gamma^s)}_{(\Gamma^{s'})}([h_1\ldots h_{\nu+1}](\lambda_1\ldots\lambda_\nu))\\
&\qquad\times\sum_{(h^{s'})(\lambda')}(\langle[\lambda_1\ldots\lambda_\nu](\lambda'),[h_1^{s'}\ldots h_{\nu+1}^{s'}](h^{s'})|[h_1\ldots h_{\nu+1}]\max;(\gamma^{s'})\rangle\\
&\qquad\times\langle a_{i\alpha}|[\lambda_1\ldots\lambda_\nu](\lambda');(\lambda_1\ldots\lambda_\nu)\max\rangle\\
&\qquad\times\langle w_{ij}|[h_1^{s'}\ldots h_{\nu+1}^{s'}](h^{s'});(0)\max\rangle)\bigg].
\end{aligned}\tag{A2}
$$

On the left-hand side of equation (A2), the product of functions in $a_{i\alpha}$ is of highest weight, equal to $\lambda_1,\ldots,\lambda_\nu$, with respect to $O(n)$. It can therefore be expanded in terms of HWS of equivalent irreps $(\lambda_1\ldots\lambda_\nu)$ of $O(n)$. Since these HWS only depend upon the $a_{i\alpha}$ variables, they are also characterised by the irrep $[\lambda_1 \ldots \lambda_\nu]$ of $U(\nu+1)$. By using appropriate $U(\nu+1)$ Wigner coefficients, they will correspond to a definite Gel'fand pattern (λ'). We therefore obtain

$$\begin{aligned}
&\langle a_{i\alpha}|[2^\nu](\lambda_0); (2^\nu)\max\rangle\langle a_{i\alpha}|[\lambda_1-2,\ldots,\lambda_\nu-2](\lambda); (\lambda_1-2,\ldots,\lambda_\nu-2)\max\rangle \\
&\quad = \sum_{(\lambda')} \langle[\lambda_1-2,\ldots,\lambda_\nu-2](\lambda),[2^\nu](\lambda_0)|[\lambda_1\ldots\lambda_\nu](\lambda');(\max)\rangle \\
&\quad\quad \times\langle a_{i\alpha}|[\lambda_1\ldots\lambda_\nu](\lambda'); (\lambda_1\ldots\lambda_\nu)\max\rangle
\end{aligned} \tag{A3}$$

The same procedure applied to the product of functions in w_{ij} in equation (A2) leads to the following relation

$$\begin{aligned}
&\langle w_{ij}|[2](h_0^s); (0)\max\rangle\langle w_{ij}|[h_1^s\ldots h_{\nu+1}^s](h^s); (0)\max\rangle \\
&\quad = \sum_{h_1^{s'}\ldots h_{\nu+1}^{s'}} \sum_{(h^{s'})} \langle[h_1^s\ldots h_{\nu+1}^s](h^s),[2](h_0^s)|[h_1^{s'}\ldots h_{\nu+1}^{s'}](h^{s'}); (\gamma_0^s)\rangle \\
&\quad\quad \times\langle w_{ij}|[h_1^{s'}\ldots h_{\nu+1}^{s'}](h^{s'}); (0)\max\rangle,
\end{aligned} \tag{A4}$$

where we have an additional summation over the $U(\nu+1)$ irrep, and

$$\begin{aligned}
(\gamma_0^s)_{ij} &= \sum_{k=1}^{j} (h_k^{s'} - h_k^s) && \text{if } i=1, \\
&= 0 && \text{if } i=2,\ldots,j,
\end{aligned} \tag{A5}$$

for $j=1,\ldots,\nu$.

When introducing equations (A3) and (A4) into the left-hand side of equation (A2), we obtain on both sides a linear combination of the products

$$\langle a_{i\alpha}|[\lambda_1\ldots\lambda_\nu](\lambda'); (\lambda_1\ldots\lambda_\nu)\max\rangle\langle w_{ij}|[h_1^{s'}\ldots h_{\nu+1}^{s'}](h^{s'}); (0)\max\rangle,$$

which are linearly independent. By equating their coefficients and using the orthogonality relations of Wigner coefficients to transfer to the left-hand side the Wigner coefficient appearing on the right-hand one, we obtain

$$\begin{aligned}
&A_{(\Gamma^{s'})}^{(\Gamma^s)}([h_1\ldots h_{\nu+1}](\lambda_1\ldots\lambda_\nu)) \\
&\quad = \sum_{(h_0^s)(\lambda_0)(h^s)(\lambda)(h^{s'})(\lambda')} \{\langle[\lambda_1-2,\ldots,\lambda_\nu-2](\lambda),[2^\nu](\lambda_0) \\
&\quad\quad \times|[\lambda_1\ldots\lambda_\nu](\lambda'); (\max)\rangle \\
&\quad\quad \times\langle[h_1^s\ldots h_{\nu+1}^s](h^s),[2](h_0^s)|[h_1^{s'}\ldots h_{\nu+1}^{s'}](h^{s'}); (\gamma_0^s)\rangle \\
&\quad\quad \times\langle[\lambda_1\ldots\lambda_\nu](\lambda'),[h_1^{s'}\ldots h_{\nu+1}^{s'}](h^{s'})|[h_1\ldots h_{\nu+1}]\max; (\gamma^{s'})\rangle \\
&\quad\quad \times\langle[\lambda_1-2,\ldots,\lambda_\nu-2](\lambda),[h_1^s,\ldots,h_{\nu+1}^s](h^s) \\
&\quad\quad \times|[h_1-2,\ldots,h_{\nu+1}-2]\max; (\gamma^s)\rangle \\
&\quad\quad \times\langle[2^\nu](\lambda_0),[2](h_0^s)|[2^{\nu+1}]\max; (\min)\rangle\}.
\end{aligned} \tag{A6}$$

On the right-hand side of equation (A6), we finally introduce the Wigner coefficient $\langle[h_1-2,\ldots,h_{\nu+1}-2]\max,[2^{\nu+1}]\max|[h_1\ldots h_{\nu+1}]\max;(\max)\rangle$, which is equal to one. The result is the expansion in Wigner coefficients terms of the recoupling coefficient, contained in the right-hand side of equation (5.6). In the latter all those operator patterns which do not contain any essential information are dropped.

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